Stochastic Processes and Their Applications 4 (1976) 243–252


# SPECIFYING ABSORPTION PROBABILITIES FOR THE SIMPLE RANDOM WALK

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Received 24 July 1973
Revised 19 January 1976

In the theory of the simple random walk an important problem is the determination of absorption probabilities. This paper considers the inverse problem in which for specified absorption probabilities, feasible regions are determined for the set of possible initial probability vectors which could generate such probabilities.

Random walk
reducible Markov chains
generalised inverse
absorption probabilities
fundamental matrix inverse

## 1. Introduction

Stochastic models in many areas such as economics, population genetics, particle physics, biology and demography give rise to multitype branching processes. These form a special class of Markov processes but we shall only be concerned here with a sub-class of these, namely discrete time reducible Markov chains consisting of one set of transient states and one set of absorbing states both these sets being of finite dimensions. It is known (see for example, [1]) that for a reducible Markov chain we have

$$x(k+1) = x(k)A\ , \tag{1}$$

$$A = \begin{bmatrix} I & 0 \\ U & T \end{bmatrix}, \tag{2}$$

where $x(k)$ is an $m + n$ row vector of state probabilities at time $k$, $T$ is the $m \times m$ matrix of transition probabilities among the $m$ transient states, $I$

the $n \times n$ unit matrix of transition probabilities among the absorbing states, and $U$ the $m \times n$ matrix of transition probabilities from the $m$ transient states to the $n$ absorbing states. Under certain conditions (cf. [3]) an equilibrium probability vector exists as $k \to \infty$. When $x$ is partitioned into $(g, e)$, then in the limit $e_\infty = 0$ and $g_\infty = e_0(I - T)^{-1} U$ where $e_0$ is the given initial probability vector of the transient states at zero time; $g_\infty$ is referred to as the vector of absorption probabilities.

This paper is concerned with the inverse problem of specifying a vector $g_\infty$ and asking which feasible region in the $m$-space of all $\{e_0\}$ vectors could have given rise to it. In order to illustrate the general theory (see Ray and Margo [6]) we shall consider the particular case of a simple random walk on the integers with two absorbing barriers. Such a treatment is of some significance because of the fundamental role played by the simple random walk in the theory of diffusion processes.

Even though the known theory describes the existence and the method of derivation of the absorption probability vector $g_\infty$, a different approach using vector $z$-transforms appears to be of some interest and is therefore given here. If we take vector $z$-transforms of the recursive equation (1) we obtain

$$\begin{aligned} z(E(z) - e_0) &= E(z)T \\ z(G(z) - g_0) &= G(z) + E(z)U . \end{aligned} \tag{3}$$

Now assuming $e_0 \neq 0$, $g_0 = 0$ and solving (3) for $E(z)$ we obtain

$$\begin{aligned} E(z) &= ze_0(zI - T)^{-1} \\ G(z) &= z(z-1)^{-1} e_0(zI - T)^{-1} U . \end{aligned}$$

Using the Tauberian theorem

$$\lim_{z \to 1} (z-1) E(z) = \lim_{k \to \infty} e(k)$$

and assuming that the limits exist we obtain

$$\lim_{k \to \infty} g(k) = g_\infty = e_0(I - T)^{-1} U . \tag{4}$$

We note that since

$$\sum_{i=1}^{m} e_{0i} = 1, \qquad \sum_{j=1}^{n} g_{\infty j} = 1 ,$$

the relation (4) defines a mapping of a $m-1$ simplex in $m$-space into a $(n-1)$ simplex in $n$-space.

## 2. The simple random walk

Now consider the random walk on the integers, with absorbing barriers $-b$ and $+a$, and with probability $p$ of moving positively, $q$ of moving negatively and $1-p-q$ of not moving ($p+q=\alpha<1$). With $m=a+b-1$ transient states $-b+1, \ldots, 0, \ldots, a-1$, and $n=2$ absorbing states $-b$, $a$, equations (1)–(2) become

$$(g(k+1), e(k+1)) = (g(k), e(k)) \begin{array}{c@{}c} \begin{array}{ccccccc} -b & a & -b+1 & \cdot & \cdot & \cdot & a-1 \end{array} & \\ \left[\begin{array}{ccccccc} 1 & 0 & & & 0 & & \\ 0 & 1 & & & & & \\ q & 0 & 1-\alpha & p & & & \\ & & q & 1-\alpha & 0 & & \\ & \cdot & & \cdot & \cdot & & \\ & \cdot & & & \cdot & \cdot & \\ & \cdot & & & \cdot & \cdot & \\ & & 0 & & & & p \\ 0 & p & & & & q & 1-\alpha \end{array}\right] & \begin{array}{c} -b \\ a \\ -b+1 \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ a-1 \end{array} \end{array}$$

While the absorption probabilities are usually obtained by the use of generating functions we prefer to obtain them here directly. To do this we need to find the inverse of $I-T$ which is not readily available in the literature on either matrix theory or probability theory (see, however, Roy and Sarhan [7] and Graybill [2]). Let $(I-T)^{-1}=M=(m_{ij})$, so that

$$M = \begin{bmatrix} \alpha & -p & \cdot & \cdot & \cdot & \\ -q & \alpha & & & 0 & \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ 0 & & & & & -p \\ & & & & -q & \alpha \end{bmatrix}^{-1}$$

By direct verification or by induction on $m$ we obtain for $j \geqslant i$

$$m_{ij} = p^{j-i}(p^i - q^i)(p^{m+1-j} - q^{m+1-j})/(p-q)(p^{m+1} - q^{m+1}),$$

$$m_{ji} = m_{ij}(q/p)^{j-i}, \tag{5a}$$

if $p \neq q$, and

$$m_{ij} = (i/p)(1 - (j/(m+1)), \tag{5b}$$

if $p = q$. The proof is omitted. From (4) and (5) we obtain

$$g_\infty = Ke_0 \begin{bmatrix} q(p^m - q^m) & p^m(p-q) \\ q^2(p^{m-1} - q^{m-1}) & p^{m-1}(p^2 - q^2) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ q^{m-1}(p^2 - q^2) & p^2(p^{m-1} - q^{m-1}) \\ q^m(p-q) & p(p^m - q^m) \end{bmatrix} \quad \text{if } p \neq q \tag{6a}$$

$$= e_0 \begin{bmatrix} 1 - 1/(m+1) & 1/(m+1) \\ 1 - 2/(m+1) & 2/(m+1) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 - m/(m+1) & m/(m+1) \end{bmatrix} \quad \text{if } p = q \tag{6b}$$

where $K = (p^{m+1} - q^{m+1})^{-1}$. If the walk begins $r$ integer steps ($r = 1, ..., a + b - 1$) from the lower boundary $-b$, then in the initial vector $e_0$ we put unit probability mass in the $r^{\text{th}}$ position. Thus $e_0 = (0, ... 0, 1, 0 ... 0)$ and the above results reduce to

$$g_{\infty 1} = \begin{cases} q^r(p^{m+1-r} - q^{m+1-r})/(p^{m+1} - q^{m+1}), & \text{if } p \neq q \quad (7a) \\ 1 - r/(m+1) & \text{if } p = q \quad (7b) \end{cases}$$

for the probability of absorption at $-b$ starting in the $r^{\text{th}}$ position on the transient set of integer states. If the walk commences at the origin then

$r = b$ and

$$g_{\infty 1} = \begin{cases} q^b(p^a - q^a)/(p^{a+b} - q^{a+b}) & \text{if } p \neq q, \\ a/(a+b) & \text{if } p = q. \end{cases}$$

It is easy to prove that the maximum and minimum probabilities of absorption at $-b(p > q)$ are given respectively by $q(p^m - q^m)/(p^{m+1} - q^{m+1})$ and $q^m(p-q)/(p^{m+1} - q^{m+1})$ for $p \neq q$, and $1 - 1/(m+1)$ and $1 - m/(m+1)$ for $p = q$. Furthermore, any initial probability distribution may be used with equations (6a) or (6b) to give the absorption probabilities; for example, if the walk has equal chance of starting in any position then $e_0 = m^{-1}(1, 1, ..., 1)$. This aspect has not so far been considered in the literature. We note that the $m$ vertices of the $m$-dimensional simplex satisfying $\Sigma_{i=1}^m e_{0i} = 1$ map onto $m$ points at and between the maximum and minimum values of $\{g_{\infty 1}\}$, the maximum and minimum corresponding to the vertices $e_0 = (1, 0, ... 0)$ and $e_0 = (0, 0, ... 0, 1)$, a result that is intuitively obvious. Also it is impossible to obtain a probability of absorption at or outside these limits and thus we have a feasible region for $\{g_{\infty 1}\}$. Fairly obvious modifications to the above may be made if we are interested in $g_{\infty 2}$ or if $p < q$.

We are now in a position to postulate an absorption probability within the feasible region and determine the corresponding feasible region for the initial probability vectors $\{e_0\}$.

## 3. Feasible region for $\{e_0\}$

The relation (4) shows that to find $\{e_0\}$ for a given value of the vector $g_\infty$ (say $g_\infty^*$) we need to compute the inverse of the matrix $U$. However, since $U$ is rectangular we have to consider its generalised inverse. We know that for the vector equation

$$xB = y,$$

where $B$ is a rectangular or singular matrix a general solution is given by

$$x = yB^- + z(BB^- - I),$$

where $z$ is an arbitrary vector of appropriate dimensions, and $B^-$ satisfying $BB^-B = B$ is the generalised inverse of $B$ (see Rao [5], Pringle and

Rayner [4]). Using this result for equation (4) we obtain

$$e_0 = (g_\infty U^- + z(UU^- - I))(I - T) . \tag{8}$$

For given $g_\infty = g_\infty^*$ equations (8) are the parametric equations of a $(m - n)$-dimensional subspace in the hyperplane $\Sigma_{i=1}^m e_{0i} = 1$ and we are required to determine its intersection with the simplex formed by this hyperplane and the positive orthant of $m$-space since we must also have $1 \geq e_{0i} \geq 0$ for $i = 1, \ldots m$. A generalised inverse $U^-$ is given by

$$U^- = \begin{bmatrix} 1/q & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1/p \end{bmatrix}.$$

We thus find that

$$e_0' = \begin{bmatrix} \alpha g_1/q \\ -pg_1/q \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ -qg_2/p \\ \alpha g_2/p \end{bmatrix} + \begin{bmatrix} qz_2 \\ -\alpha z_2 + qz_3 \\ pz_2 - \alpha z_3 + qz_4 \\ \cdot \\ \cdot \\ \cdot \\ pz_{n-3} - \alpha z_{n-2} + qz_{n-1} \\ -\alpha z_{n-2} + qz_{n-1} \\ pz_{n-1} \end{bmatrix} \tag{9}$$

Now we require (given $g_\infty = g_\infty^*$) that this $(m - 2)$ sub-space intersect the $(m - 1)$-simplex. Since the boundaries of this simplex are of dimension 0, 1, ..., $\{m - 2\}$, and also lie in the coordinate planes, the $(m - 2)$-subspace will intersect the 1-space boundaries of the simplex in 0-space, i.e. points. This is clear from the result that in $m$ dimensions a $p$-space intersects a $q$-space in a $(p + q - m)$-space (see for example Somerville [8]). Since there are ${}^mC_2$ 1-space boundaries of the simplex we obtain these many vertices, some of which lie outside the positive orthant leaving the remainder to define the feasible region for $\{e_0\}$. In practice the procedure is the following: (i) put any $(m - 2)$ $e_{0i}$'s equal to zero in equation (9) and solve for the $(m - 2)$ unknown $z$'s, and (ii) substitute these $z$ values into the remaining two equations to obtain the 2 remaining $e_{0i}$'s. This then gives the co-ordinates of a vertex on the inter-

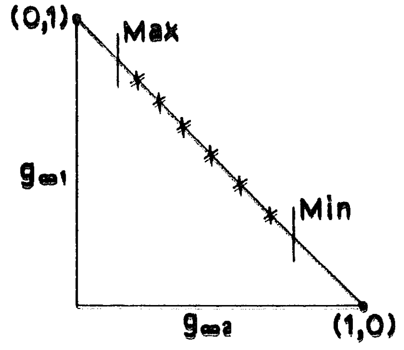


Fig. 1. Feasible region in $g$-space.

section of the $(m-2)$ subspace with the $(m-1)$-simplex. The procedure repeated ${}^mC_2$ times gives all vertices and the ones with all positive or zero coordinates are those which define the feasible region. Since we are dealing only with linear transformations and we begin with a convex region (i.e. a $(m-1)$-simplex), this feasible region will in general be a convex polytope.

In order to illustrate the foregoing theory we shall consider two simple examples with $m = 3$, and note the geometrical consequences. In this case equation (9) becomes

$$e_0' = \begin{bmatrix} \alpha & -q & 0 \\ -p & \alpha & -q \\ 0 & -p & \alpha \end{bmatrix} \begin{bmatrix} g_1/q \\ -z_2 \\ g_2/p \end{bmatrix} .$$

Putting $e_{0i} = e_i$ $(i = 1, 2, 3)$ and specifying $g_1 = g_1^*$, we can re-write these equations as

$$\frac{e_1 - \alpha g_1^*/q}{q} = \frac{e_2 + (pg_1^*/q + qg_2^*/p)}{-\alpha} = \frac{e_3 - \alpha g_2/p}{p} = z_2 .$$

Thus for different $g_1^*$ we obtain different feasible regions, which are the intersection of these parallel lines with the hyperplane $e_1 + e_2 + e_3 = 1$ in the positive quadrant (see Fig. 1). For simplicity let us take $p = q$ so that

$$\frac{e_1 - 2g_1^*}{1} = \frac{e_2 + 1}{-2} = \frac{e_3 - 2g_2^*}{1} = pz . \tag{10}$$

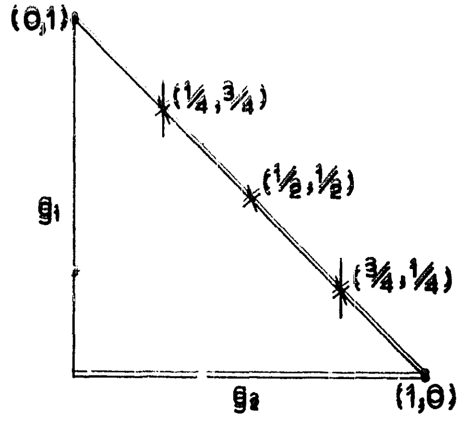


Fig. 2. Feasible regions in $e$-space for Examples 1 and 2.

The ${}^3C_2 = 3$ possible vertices are obtained by setting $e_1$, $e_2$, $e_3$ equal to zero in turn in (10). Thus

$$
\begin{aligned}
e_1 &= 0 \quad : \quad e_2 = 4g_1^* - 1, \quad e_3 = 2 - 4g_1^* \,, \\
e_2 &= 0 \quad : \quad e_1 = -\tfrac{1}{2} + 2g_1^*, \quad e_3 = 3/2 - 2g_1^*, \\
e_3 &= 0 \quad : \quad e_1 = -2 + 4g_1^*, \quad e_2 = 3 - 4g_1^* \,.
\end{aligned}
\tag{11}
$$

We recall that the maximum and minimum values of $g_{\infty 1}$ are given by $1 - 1/(m+1) = \frac{3}{4}$ and $1 - m/(m+1) = \frac{1}{4}$. Hence we may only choose a $g_1^*$ in the range $(\frac{1}{4}, \frac{3}{4})$. A $g_1^* = 7/8$ would not be feasible, that is, it could not have arisen from a valid initial probability vector over the three transient states.

**Example 1.** If $g_1^* = 2/3$ the three vertices in (11) become

$$
\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5/3 \\ -2/3 \end{bmatrix}, \quad \begin{bmatrix} 5/6 \\ 0 \\ 1/6 \end{bmatrix}, \quad \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \end{bmatrix}.
$$

It will be seen that the first vertex lies outside the positive quadrant and is therefore infeasible. Thus any point on the line joining the points (5/6, 0, 1/6) and (2/3, 1/3, 0) may represent the initial probabilities and give rise to a probability of absorption at $a$ of 2/3 (see Fig. 2).

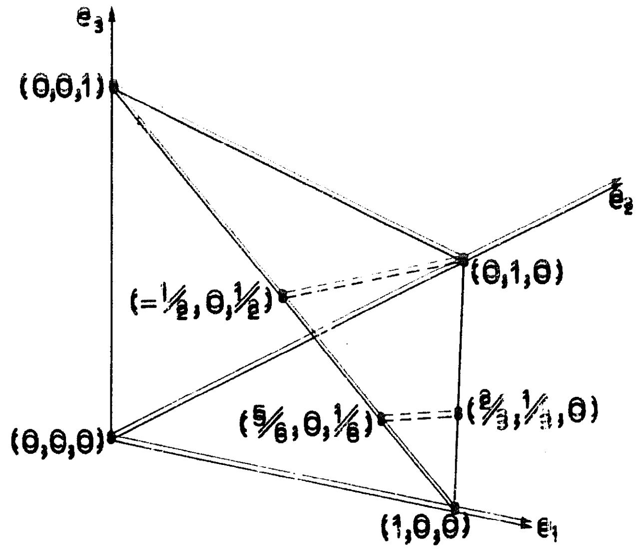


Fig. 3. Feasible regions in $e$-space for Examples 1 and 2.

**Example 2.** For $g_1^* = \frac{1}{2}$ the possible vertices are

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and these are situated on the line

$$\frac{e_1 - 1}{1} = \frac{e_2 + 1}{-2} = \frac{e_3 - 1}{1} = pz .$$

Hence we may choose $e_2$ anywhere in the range $(0, 1)$ but with $e_1 = e_3 = (1 - e_2)/2$. This is what we should expect intuitively, since by symmetry for desired equal probabilities of absorption at $a$ and $-b$ we should place any probability mass at the origin and share the remaining probability mass equally on the $+1$ and $-1$ states.

## References


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